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The effect of trapped field on stationary waves formed at the interface of a conduction fluid and a magnetic field

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A two-dimensional, moving, incompressible and electrically conducting fluid having trapped magnetic field (i.e. field threading the fluid) is confined by a uniform vacuum magnetic field. The fluid-vacuum field interface is horizontal and assumed to be free from instabilities. Dissipation by viscosity and resistivity is neglected, hence, for steady motion, the trapped field may be regarded as frozen into the fluid, in the sense that both move together. A pressure disturbance is introduced into the system, and it is found that a harmonic stationary wave is formed upstream provided certain criteria hold.

1. Introduction

Essentially the problem under investigation is the magnetohydrodynamic analogue of the classical surface wave problem in hydrodynamics. Figure 1 shows an infinitely conducting and incompressible fluid flowing down a channel of

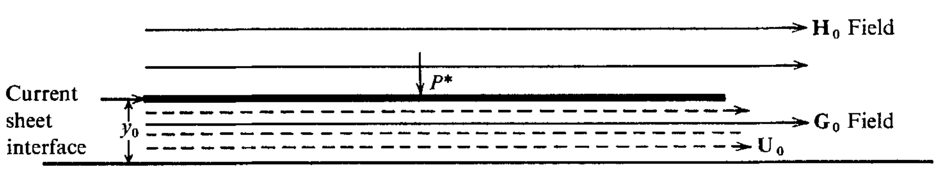

FIGURE 1. A column of conducting fluid moving with velocity $\mathbf{U}_0$ in the presence of trapped field $\mathbf{G}_0$ and confined by a vacuum field $\mathbf{H}_0$.

uniform depth y_0. The fluid, which is assumed to be inviscid, is threaded by a trapped magnetic field G_0, and confined by a vacuum field H_0. All electric current is confined to a very thin layer, which forms the vacuum field—conducting fluid interface.

This initially uniform system can be disturbed in one of two ways. It is possible to affect the vacuum field by introducing into it either a current source or a magnetic dipole. Alternatively, a pressure force p^*, due to a jet of gas having no interaction with the magnetic field, can be applied at some point on the interface. For mathematical convenience we choose to adopt the latter method, and it is

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found that a stationary surface wave is formed upstream of the source of disturbance provided the following conditions hold:

$$\beta + \beta_1 > 1,$$
$$\beta_1 < 1,$$

where $\beta = (\mu_0 H_0^2)/(\rho_0 U_0^2)$ is an Alfvén number based on the vacuum field H_0, $\beta_1 = (\mu_1 G_0^2)/(\rho_0 U_0^2)$ is an Alfvén number based on the trapped field G_0. ρ_0, U_0 are the initial uniform values of density and velocity and μ_0, μ_1 are respectively the permeabilities of free space and the fluid.

The physical significance of these conditions is discussed in detail in a later section; but simple inspection shows that in general the presence of trapped magnetic field plays an important role in determining whether a wave can propagate. For the particular case of no trapped field, $\beta_1 = 0$, hence $\beta > 1$ becomes a sufficient condition for an upstream wave. This is not an unexpected result, since this is precisely the condition necessary for an upstream wake to be formed when an infinitely conducting and incompressible fluid flows past an obstacle in the presence of an aligned magnetic field (Goldsworthy 1961).

2. Method of solution

The initial two-dimensional configuration is shown in figure 1. The base of the channel is taken to be the x axis, whilst the y axis is perpendicular through the line of applied pressure p^*. The undisturbed fluid pressure is P_0, hence the balance of fluid and magnetic pressures across the interface at $y = y_0$ yields;

$$P_0 + \frac{\mu_1 G_0^2}{2} = \frac{\mu_0 H_0^2}{2}; \tag{2.1}$$

therefore

$$\frac{P_0}{\frac{1}{2}\rho_0 U_0^2} + \beta_1 = \beta, \tag{2.2}$$

which shows that the initial fluid pressure is determined according to the relative strengths of the trapped and vacuum magnetic fields.

On application of a pressure disturbance p^*, the interface assumes the shape $y = y_0 + \eta(x)$ whilst the vacuum field becomes $\mathbf{H} = \mathbf{H}_0 + \mathbf{h}$, the trapped field $\mathbf{G} = \mathbf{G}_0 + \mathbf{g}$, the fluid velocity $\mathbf{U} = \mathbf{U}_0 + \mathbf{q}$ and the fluid pressure $P' = P_0 + p_F$, where it is assumed that the quantities $\eta(x)$, p_F, $\mathbf{q} = (u, v)$, $\mathbf{h} = (h_x, h_y)$, $\mathbf{g} = (g_x, g_y)$ are perturbations whose squares and products can be neglected. Since all current is confined to the interface, $\mathbf{h}$ satisfies the equations, $\operatorname{curl} \mathbf{h} = \operatorname{div} \mathbf{h} = 0$; hence $\mathbf{h}$ is expressible in terms of a magnetic scalar potential which satisfies Laplace's equation:

$$\mathbf{h} = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}\right), \quad \text{where} \quad \nabla^2 \phi = 0. \tag{2.3}$$

The steady fluid motion is governed by the following equations:

$$\operatorname{div} \mathbf{U} = 0, \quad \operatorname{div} \mathbf{G} = 0, \tag{2.4}$$

$$\rho \mathbf{U} . \nabla \mathbf{U} = -\nabla P' + \mu_1 \operatorname{curl} \mathbf{G} \times \mathbf{G}, \tag{2.5}$$

$$\operatorname{curl} (\mathbf{U} \times \mathbf{G}) = 0, \tag{2.6}$$

which, when linearized, become:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0, \tag{2.7}$$

$$U_0\frac{\partial u}{\partial x}-\frac{\mu_1}{\rho_0}G_0\frac{\partial g_x}{\partial x}+\frac{\partial p}{\partial x} = 0, \tag{2.8}$$

$$U_0\frac{\partial v}{\partial x}-\frac{\mu_1}{\rho_0}G_0\frac{\partial g_y}{\partial x}+\frac{\partial p}{\partial y} = 0, \tag{2.9}$$

$$U_0\frac{\partial g_y}{\partial x}-G_0\frac{\partial v}{\partial x} = 0, \tag{2.10}$$

$$\frac{\partial g_x}{\partial x}+\frac{\partial g_y}{\partial y} = 0, \tag{2.11}$$

where $\rho_0 p = p_f+\mu_1 G_0 g_x$ is a perturbation of the total pressure (magnetic and fluid).

Boundary conditions

There is no disturbance at infinity, hence,

$$\phi \to 0 \quad \text{as} \quad y \to \infty. \tag{2.12}$$

On the channel base, the assumption of an inviscid and infinitely conducting fluid yields

$$v = g_y = 0 \quad \text{on} \quad y = 0. \tag{2.13}$$

At the interface, $y = y_0+\eta$, which is both a streamline and a magnetic field line, then approximately,

$$\frac{\partial \eta}{\partial x} = \frac{h_y}{H_0} = \frac{g_y}{G_0}. \tag{2.14}$$

Finally, the pressure balance at the interface yields,

$$P'+\frac{\mu_1}{2}G^2 = \mu_0\frac{H^2}{2}+p^*$$

which, in view of (2.1), becomes

$$\rho_0 p = p_f+\mu_1 G_0 g_x = \mu_0 H_0 h_x+p^*. \tag{2.15}$$

It can readily be shown that (2.7)–(2.11) have the following solution:

$$\left.\begin{aligned} u = U_0\Psi_x,\quad g_x = G_0\Psi_x,\\ v = U_0\Psi_y,\quad g_y = G_0\Psi_y,\end{aligned}\quad p = \left(\frac{\mu_1}{\rho_0}G_0^2-U_0^2\right)\Psi_x,\right\}$$

where Ψ is a solution of $\nabla^2\Psi = 0$. (2.16)

Expressing ϕ and Ψ as Fourier transforms:

$$\phi = \frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{\phi}\,e^{i\alpha x}\,d\alpha,\quad \Psi = \frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{\Psi}\,e^{i\alpha x}\,d\alpha,$$

then (2.3) and (2.16) yield

$$\bar{\Psi} = A_1 e^{-|\alpha|y}+B_1 e^{|\alpha|y},\quad \bar{\phi} = C e^{-|\alpha|y}+D e^{|\alpha|y},$$

and the boundary conditions (2.12) and (2.13) give

$$\bar{\Psi} = 2A \cosh|\alpha|y, \quad \bar{\phi} = Ce^{-|\alpha|y}.$$

If, for convenience, we choose our pressure disturbance of the form,

$$p^* = \frac{P}{2\pi}\frac{b}{b^2+x^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tfrac{1}{2}P e^{-|\alpha|b} e^{i\alpha x}\, d\alpha,$$

then the boundary conditions (2.14) and (2.15), applicable on $y = y_0$, enable us to solve the surface displacement:

$$\eta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\eta}\, e^{i\alpha x} d\alpha = -\frac{P}{4\pi\rho_0}\int_{-\infty}^{\infty} \frac{e^{i\alpha x - |\alpha| b}\, d\alpha}{|\alpha|[\beta-(1-\beta_1)\coth|\alpha|y_0]}. \tag{2.17}$$

Examination of the integrand of (2.17) shows that there are poles on the real axis where $\coth|\alpha|y_0 = \beta/(1-\beta_1)$, and a branch point at the origin. (At first glance there seems to be a pole at the origin, yet close inspection shows that this is not so.) Since $\coth|\alpha|y_0 \geqslant 1$, necessary and sufficient conditions for poles to exist are

$$\beta+\beta_1 \geqslant 1, \tag{2.18}$$

$$\beta_1 < 1. \tag{2.19}$$

With the above conditions satisfied, we can proceed to evaluate the integral in (2.17) by contour integration after using a radiation condition, following a method due to Lighthill (1960), which effectively removes the poles from the real α axis, and, in this problem, into the lower half plane. The following results are then obtained:

for $x < 0$,

$$\eta(x) = \frac{k4\pi e^{-cb}}{cy_0(\coth^2 cy_0 - 1)}\sin cx + 2k\int_0^{\infty} \frac{e^{px}}{p}\frac{[\beta/(1-\beta_1)\cos bp - \cot py_0 \sin bp]}{[\cot^2 py_0 + \beta/(1-\beta_1)^2]}\, dp$$

for $x > 0$, $$\eta(x) = 2k\int_0^{\infty} \frac{e^{-px}}{p}\frac{[\beta/(1-\beta_1)\cos bp - \cot py_0 \sin bp]}{[\cot^2 py_0 + (\beta/(1-\beta_1))^2]}\, dp, \tag{2.20}$$

where $$k = \frac{-P}{4\pi\rho_0(1-\beta_1)}, \quad c = \frac{1}{y_0}\coth^{-1}\left(\frac{\beta}{1-\beta_1}\right). \tag{2.21}$$

So, provided (2.18) and (2.19) hold, a stationary harmonic wave having wave number c is formed upstream of the pressure disturbance, whilst a local disturbance corresponding to the integral terms in (2.20) is formed symmetrically about the origin. This latter effect dies out rapidly with increasing distance from the origin due to the presence of the exponential factor, $e^{-p|x|}$ appearing in each integrand.

In the light of the above results, we can examine the special or limiting case of condition (2.18). When $\beta+\beta_1 = 1$ the integrand of (2.17) has poles on the real axis; however, $\coth cy_0$ now becomes unity and consequently the surface wave has infinite amplitude. This situation must be avoided for linearization to be valid, hence necessary and sufficient criteria for an upstream wave are

$$\beta+\beta_1 > 1, \tag{2.22}$$

$$\beta_1 < 1. \tag{2.23}$$

3. Discussion of results

First we shall examine the physical significance of the two criteria which are necessary for a harmonic wave to be formed.

It is well known in magnetohydrodynamics that a disturbance to an incompressible and electrically conducting fluid of infinite extent can only propagate upstream if the upstream pull (tension) of the magnetic field exceeds the downstream momentum of the fluid, or, equivalently, if the Alfvén velocity is greater than the fluid velocity. Condition (2.22) says precisely this, that on the interface, along which the disturbance is convected, the combined tension of the vacuum and trapped fields must exceed the downstream momentum of the fluid.

It is not directly obvious that condition (2.23) should hold, therefore we shall look at this condition more closely. Condition (2.23) says that $(\beta_1 - 1) < 0$, and the only part of the analysis in which the factor $(\beta_1 - 1)$ arises is in the pressure relation at the interface, namely (2.15), which can be expressed in the form,

$$(\beta_1 - 1)\frac{g_x}{G_0} = \frac{\beta h_x}{H_0} + \frac{p^*(x)}{\rho_0 U_0^2}. \tag{3.1}$$

This condition says that well away from the origin, where the influence of $p^*(x)$ is negligible, g_x and h_x will have the same or opposite signs, according as $\beta_1 > 1$ or $\beta_1 < 1$. Thus the problem is now to show that a necessary but not sufficient condition for waves is that g_x and h_x should have opposite signs well away from the origin. If we assume that far upstream the interface is wavy in shape, then this implies that a compression of the vacuum field is accompanied by an expansion of the trapped field and *vice versa*. Furthermore, a compression of the vacuum field corresponds to an increase in magnetic pressure of the vacuum field, i.e. $h_x > 0$; whilst an expansion of the trapped field corresponds to a decrease in magnetic pressure, i.e. $g_x < 0$. Hence, in order that a wave may be formed, g_x and h_x must have opposite signs, i.e. $\beta_1 < 1$.

Conditions (2.22) and (2.23) essentially provide a range of feasible velocities, such that, if the initial fluid velocity U_0 is within this range, then a wave having phase speed U_0 is formed upstream. In particular,

$$U_a^2 < U_0^2 < U_a^2 + U_b^2, \tag{3.2}$$

where U_a and U_b are respectively the Alfvén speed in the fluid and an Alfvén speed based on the vacuum field and the fluid density.

Condition (3.2) is analogous to the criterion for gravity surface waves in hydrodynamics, $U_0^2 < gh$, where g is gravity and h is the depth of the channel.

Finally, we must discuss the existence of a steady state solution given by (2.20) and (2.21). This can be verified *via* an unsteady analysis, assuming that the pressure disturbance is 'switched on' at time $t = 0$, say, and writing $p^*(x,t) = p^*(x)\,H(t)$. Taking a Fourier transform on time with ω the transform variable, we find that

$$\bar{p}^*(\alpha, \omega) = \bar{p}^*(\alpha)\left\{\frac{\delta(\omega)}{2} + \frac{1}{2\pi i\omega}\right\};$$

and Lim (1968), treating a similar axisymmetric problem, used steepest descents to ascertain that, for large time, the solution consists of the above steady component accompanied by extra terms decaying algebraically with time.

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